Algebra: Expressions and Equations

Solving Algebraic Equations– Typical Methods and Strategies

Solving an equation that contains an unknown quantity (the *variable*) means finding all values of the variable so that, when these values are substituted for the variable, the equation is true. Some equations are easy to solve just by "looking", such as

$x + 5 = 20$

$\frac{y}{4} = -3$

$x^2 = 36$

For other equations, more algebra may be needed in order to find the solutions. Here are some typical situations and strategies.

- **An equation of the form [*expression*] = 0, where [*expression*] may be factored into two or more linear factors.** After factoring, you thus have something like $a(x-b)(x-c)\cdots(x-d) = 0$, where $a, b, c, \ldots d$ are real numbers and $a \neq 0$. Then, since *a product of numbers is zero if and only if (at least) one of the factors is zero*, you may conclude that $x - b = 0$ or $x - c = 0$ or $\ldots x - d = 0$. In other words the solutions to the equation are $x = b, x = c, \ldots, x = d$.

  example: To solve the equation $x^3 + 3x^2 - 18x = 0$, factor the left hand side as $x(x+6)(x-3)$. The solutions are thus $x = 0, x = -6$, and $x = 3$.

- A variation of the above example includes **expressions which may not factor completely into linear factors (or not easily.)** For such equations you may need to use **the quadratic formula.** While it is possible in some cases to find *complex* (or imaginary) roots– solutions involving $i = \sqrt{-1}$– in Calculus I this will rarely if ever be necessary.

example: In the equation $(x^2+2)(x^2-3x-1)=0$, the first factor has no real solutions. For the second factor, use the quadratic formula: $x=\frac{-(-3)\pm\sqrt{(-3)^2-4(-1)}}{2}$, so the solutions are $x=\frac{3}{2}+\frac{\sqrt{13}}{2}$ and $x=\frac{3}{2}-\frac{\sqrt{13}}{2}$.

- **An equation of the form [*expression*] = 0, where [*expression*] contains one or more quotients.** (If the quotients contain polynomials, they are usually called *rational functions*.) A common strategy for this type of equation is to combine the terms into a single quotient to get something like $\frac{\textit{top expression}}{\textit{bottom expression}}=0$, which, assuming the numerator and denominator have no common factors, means that *top expression* $=0$. (We are using the principle that *a (reduced) quotient is 0 if and only if the numerator is zero.* Can you see why this is true?) If necessary, the techniques described in the above two cases can then be used to find the solutions.

example: To solve the equation $\frac{x+1}{x+5}-\frac{1}{x+2}=0$, write the left hand side as the single fraction $\frac{(x+1)(x+2)-(x+5)}{(x+5)(x+2)}$. After expanding and collecting terms you get $\frac{x^2+2x-3}{(x+5)(x+2)}$. Then $x^2+2x-3=(x+3)(x-1)=0$ so the solutions are $x=-3$ and $x=1$.